# Liquid Welfare Guarantees for No-Regret Learning in Sequential Budgeted Auctions

EC 2023

**Giannis Fikioris**, Éva Tardos

Cornell University

# Sequential Budgeted Second-price Auctions

# Sequential Budgeted Second-price Auctions

- $T$ rounds
- $n$ players
- Value $v_{it} \in [0, 1]$

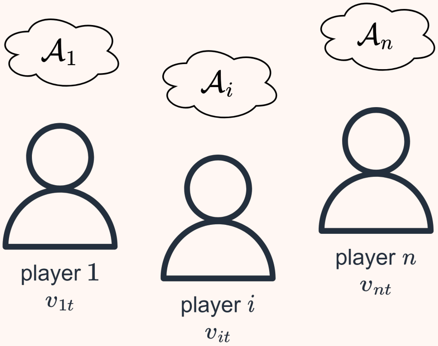
- $T$ rounds
- $n$ players
- Value $v_{it} \in [0, 1]$
- $V_i = \sum_{t:\text{won}} v_{it}$

- $T$ rounds
- $n$ players
- Value $v_{it} \in [0, 1]$
- $V_i = \sum_{t:\text{won}} v_{it}$
- Utilities are *budgeted* *quasi-linear*:

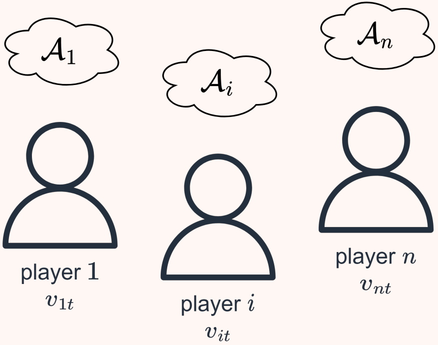
$$U_i = \begin{cases} V_i - P_i, & \text{if } P_i \leq B_i \\ -\infty, & \text{otherwise} \end{cases}$$

- $T$ rounds
- $n$ players
- Value $v_{it} \in [0, 1]$
- $V_i = \sum_{t:\text{won}} v_{it}$
- Utilities are *budgeted* *quasi-linear*:

$$U_i = \begin{cases} V_i - P_i, & \text{if } P_i \leq B_i \\ -\infty, & \text{otherwise} \end{cases}$$



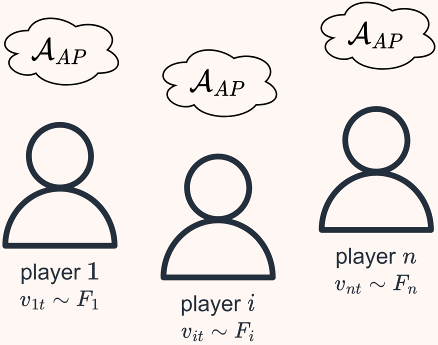
- *Adaptive Pacing* Algorithm $\mathcal{A}_{AP}$




[Balseiro-Gur, EC'17]

# Individual player guarantees

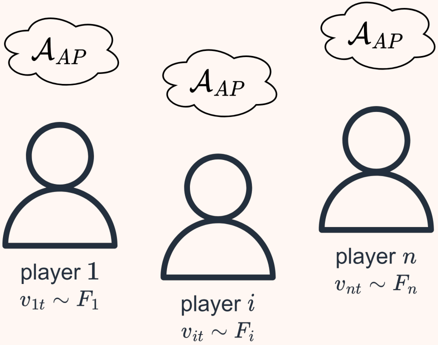
- *Adaptive Pacing* Algorithm $\mathcal{A}_{AP}$
- Cooperation guarantee:




[Balseiro-Gur, EC'17]

- *Adaptive Pacing* Algorithm $\mathcal{A}_{AP}$
- Cooperation guarantee: for all $i$

$$U_i \geq \max_{\vec{b}'_i} \left( \hat{U}_i(\vec{b}'_i) \right) - \tilde{O}(\sqrt{T})$$

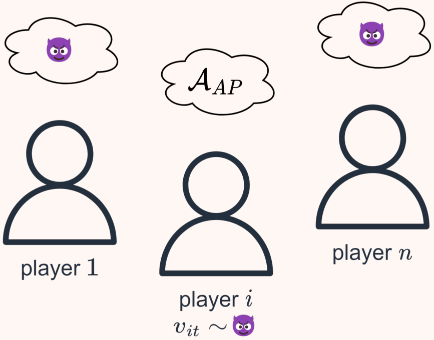



[Balseiro-Gur, EC'17]

- *Adaptive Pacing* Algorithm $\mathcal{A}_{AP}$
- Cooperation guarantee: for all $i$

$$U_i \geq \max_{\vec{b}'_i} \left( \hat{U}_i(\vec{b}'_i) \right) - \tilde{O}(\sqrt{T})$$

- Adversarial guarantee:



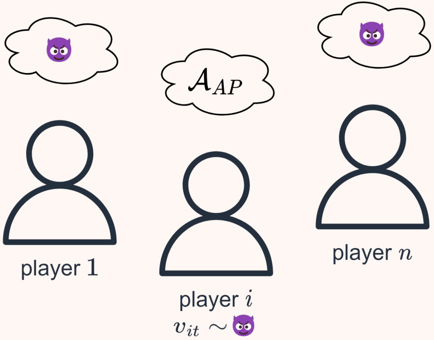

[Balseiro-Gur, EC'17]

- *Adaptive Pacing* Algorithm $\mathcal{A}_{AP}$
- Cooperation guarantee: for all $i$

$$U_i \geq \max_{\vec{b}_i'} \left(\hat{U}_i(\vec{b}_i')\right) - \tilde{O}(\sqrt{T})$$

- Adversarial guarantee: for some $\gamma \geq 1$

$$U_i \geq \frac{\max_{\vec{b}_i'} \left(\hat{U}_i(\vec{b}_i')\right) - \tilde{O}(\sqrt{T})}{\gamma}$$



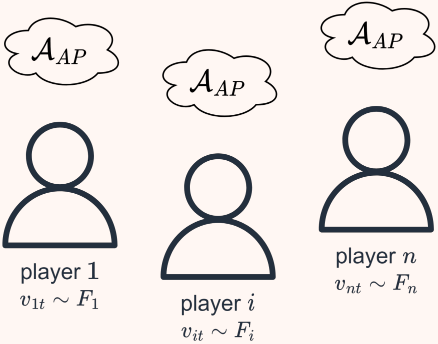

[Balseiro-Gur, EC'17]

# Aggregate Guarantee

- Liquid welfare[1]:
  - Social welfare for Budgeted settings
  - $\texttt{LW}_i = \min\{B_i, V_i\}$
  - $\texttt{LW} = \sum_i \texttt{LW}_i$

[1] [Dobzinski-Paes Leme, ICALP'14]

- Liquid welfare[1]:
  - Social welfare for Budgeted settings
  - $\texttt{LW}_i = \min\{B_i, V_i\}$
  - $\texttt{LW} = \sum_i \texttt{LW}_i$

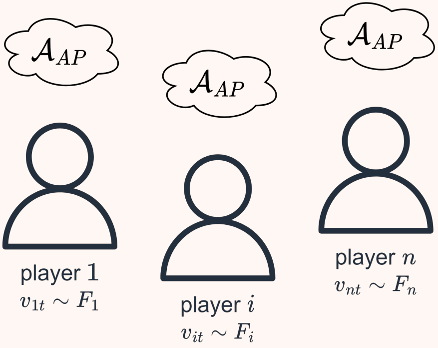



$$\forall i,\ U_i \geq \max_{\vec{b}_i'} \left(\hat{U}_i(\vec{b}_i')\right) - \tilde{O}(\sqrt{T})$$

[1][Dobzinski-Paes Leme, ICALP'14]
[2][Gaitonde-Light-Lucier-Slivkins, ITCS'23]

- Liquid welfare[1]:
  - Social welfare for Budgeted settings
  - $\mathtt{LW}_i = \min\{B_i, V_i\}$
  - $\mathtt{LW} = \sum_i \mathtt{LW}_i$
- On expectation[2]

$$\mathtt{LW} \geq \frac{\mathtt{LW}^* - \tilde{O}(n\sqrt{T})}{2}$$



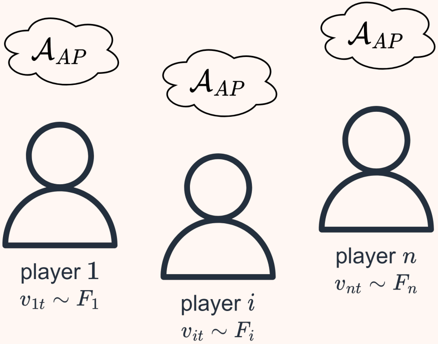

$$\forall i,\ U_i \geq \max_{\vec{b}'_i} \left(\hat{U}_i(\vec{b}'_i)\right) - \tilde{O}(\sqrt{T})$$

[1] [Dobzinski-Paes Leme, ICALP'14]
[2] [Gaitonde-Light-Lucier-Slivkins, ITCS'23]

- Liquid welfare[1]:
  - Social welfare for Budgeted settings
  - $\texttt{LW}_i = \min\{B_i, V_i\}$
  - $\texttt{LW} = \sum_i \texttt{LW}_i$
- On expectation[2]

$$\texttt{LW} \geq \frac{\texttt{LW}^* - \tilde{O}(n\sqrt{T})}{2}$$

- Everyone uses $\mathcal{A}_{AP}$




$$\forall i,\ U_i \geq \max_{\vec{b}'_i} \left(\hat{U}_i(\vec{b}'_i)\right) - \tilde{O}(\sqrt{T})$$

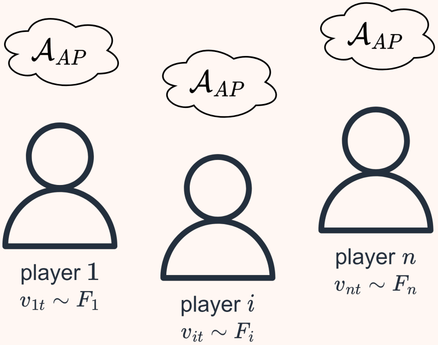
[1] [Dobzinski-Paes Leme, ICALP'14]
[2] [Gaitonde-Light-Lucier-Slivkins, ITCS'23]

# Aggregate Guarantee

- Liquid welfare[1]:
  - Social welfare for Budgeted settings
  - $\texttt{LW}_i = \min\{B_i, V_i\}$
  - $\texttt{LW} = \sum_i \texttt{LW}_i$
- On expectation[2]

$$\texttt{LW} \geq \frac{\texttt{LW}^* - \tilde{O}(n\sqrt{T})}{2}$$

- Everyone uses $\mathcal{A}_{AP}$
- Values are Bayesian



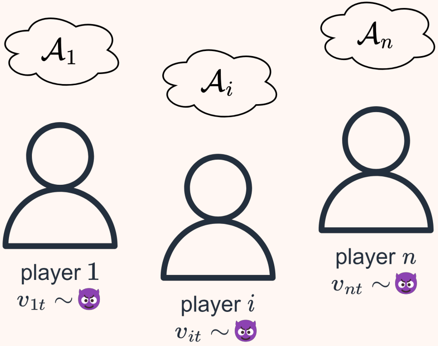

$$\forall i,\ U_i \geq \max_{\vec{b}'_i} \left(\hat{U}_i(\vec{b}'_i)\right) - \tilde{O}(\sqrt{T})$$

[1] [Dobzinski-Paes Leme, ICALP'14]
[2] [Gaitonde-Light-Lucier-Slivkins, ITCS'23]

- Do individual guarantees imply welfare guarantees?

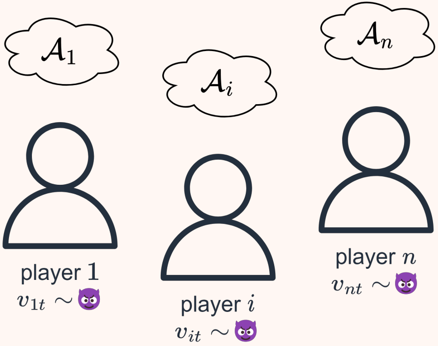
# Liquid Welfare of General Algorithms

- Do individual guarantees imply welfare guarantees?




$$\forall i,\ U_i \geq \frac{\max_{\vec{b}_i'}\left(\hat{U}_i(\vec{b}_i')\right) - \mathrm{Reg}}{\gamma}$$

- Do individual guarantees imply welfare guarantees?
- Not in second-price

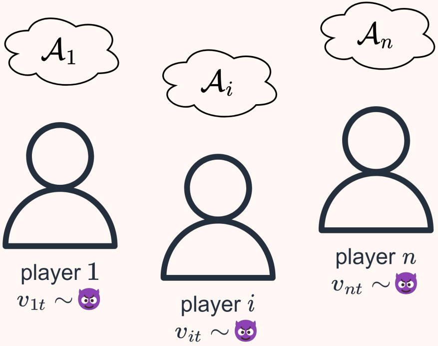



$$\forall i,\ U_i \geq \frac{\max_{\vec{b}'_i}\left(\hat{U}_i(\vec{b}'_i)\right) - \mathrm{Reg}}{\gamma}$$

[Gaitonde-Light-Lucier-Slivkins, ITCS'23]

# Liquid Welfare of General Algorithms

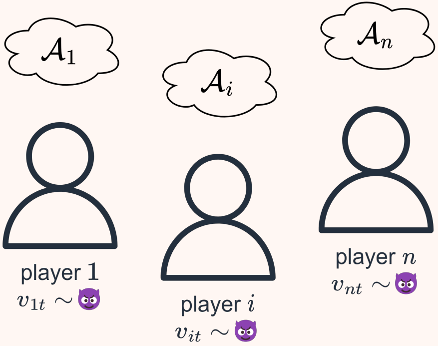
- Do individual guarantees imply welfare guarantees?
- Not in second-price
  - $n = 2$
  - $v_{1t} = v_{2t} = 1$
  - $\gamma = 1$, $\mathrm{Reg} = 0$




$$\forall i, \; U_i \geq \frac{\max_{\vec{b}_i'} \left(\hat{U}_i(\vec{b}_i')\right) - \mathrm{Reg}}{\gamma}$$

[Gaitonde-Light-Lucier-Slivkins, ITCS'23]

- Do individual guarantees imply welfare guarantees?
- Not in second-price
  - $n = 2$
  - $v_{1t} = v_{2t} = 1$
  - $\gamma = 1$, $\mathrm{Reg} = 0$
  - $\mathrm{LW} = \varepsilon \mathrm{LW}^*$




$$\forall i,\ U_i \geq \frac{\max_{\vec{b}_i'} \left(\hat{U}_i(\vec{b}_i')\right) - \mathrm{Reg}}{\gamma}$$

[Gaitonde-Light-Lucier-Slivkins, ITCS'23]

# Sequential Budgeted First-price Auctions

- Second-price: compete against

$$\max_{\vec{b}_i} \left( \hat{U}_i(\vec{b}_i) \right)$$

- Second-price: compete against

$$\max_{\vec{b}_i} \left( \hat{U}_i(\vec{b}_i) \right)$$

- Too strong for first-price, even when unbudgeted:

- Second-price: compete against

$$\max_{\vec{b}_i} \left( \hat{U}_i(\vec{b}_i) \right)$$

- Too strong for first-price, even when unbudgeted:
  - Second-price: bid value, $\mathtt{Reg} = 0$

- Second-price: compete against

$$\max_{\vec{b}_i} \left( \hat{U}_i(\vec{b}_i) \right)$$

- Too strong for first-price, even when unbudgeted:
  - Second-price: bid value, $\mathrm{Reg} = 0$
  - First-price: $\mathrm{Reg} \geq \Omega(T)$

- Best 'action' in hindsight

- Best ‘action’ in hindsight
- Shading multiplier $\lambda \in [0, 1]$: bid $\lambda v_{it}$

- Best ‘action’ in hindsight
- Shading multiplier $\lambda \in [0, 1]$: bid $\lambda v_{it}$
- Best multiplier in hindsight: $\max_{\lambda \in [0,1]} \hat{U}_i(\lambda)$

- Best 'action' in hindsight
- Shading multiplier $\lambda \in [0, 1]$: bid $\lambda v_{it}$
- Best multiplier in hindsight: $\max_{\lambda \in [0,1]} \hat{U}_i(\lambda)$

- Shading multipliers in $\mathcal{A}_{AP}$:

# Sequential Budgeted First-price Auctions – Benchmark

- Best 'action' in hindsight
- Shading multiplier $\lambda \in [0,1]$: bid $\lambda v_{it}$
- Best multiplier in hindsight: $\max_{\lambda \in [0,1]} \hat{U}_i(\lambda)$

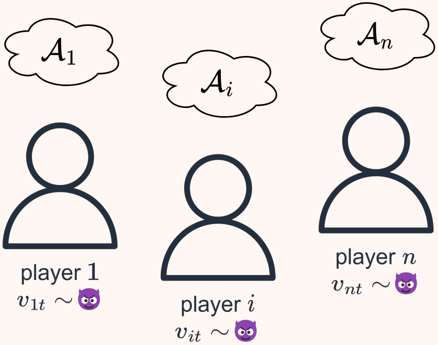
- Shading multipliers in $\mathcal{A}_{AP}$:
  - Player $i$ bids $\lambda_{it} v_{it}$

- Best 'action' in hindsight
- Shading multiplier $\lambda \in [0, 1]$: bid $\lambda v_{it}$
- Best multiplier in hindsight: $\max_{\lambda \in [0,1]} \hat{U}_i(\lambda)$

- Shading multipliers in $\mathcal{A}_{AP}$:
  - Player $i$ bids $\lambda_{it} v_{it}$
  - Computes $\lambda_{i,t+1}$ based on spending

- Best 'action' in hindsight
- Shading multiplier $\lambda \in [0, 1]$: bid $\lambda v_{it}$
- Best multiplier in hindsight: $\max_{\lambda \in [0,1]} \hat{U}_i(\lambda)$

- Shading multipliers in $\mathcal{A}_{AP}$:
  - Player $i$ bids $\lambda_{it} v_{it}$
  - Computes $\lambda_{i,t+1}$ based on spending



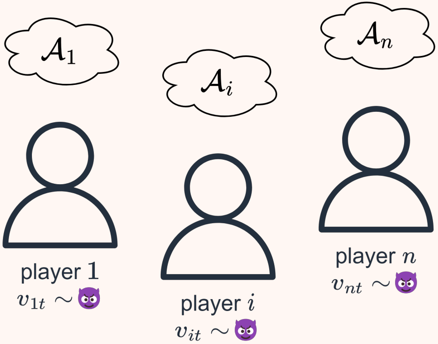

$$\forall i, \; U_i \geq \frac{\max_\lambda \hat{U}_i(\lambda) - \mathrm{Reg}}{\gamma}$$

# Guarantees in Sequential First-price Auctions

**Theorem – Liquid Welfare guarantee**

If players are $\gamma$-competitive then

$$\texttt{LW} \geq \frac{\texttt{LW}^* - O(n\texttt{Reg})}{\gamma + \frac{1}{2} + O\left(\frac{1}{\gamma}\right)}$$




$$\forall i,\ U_i \geq \frac{\max_\lambda \hat{U}_i(\lambda) - \texttt{Reg}}{\gamma}$$

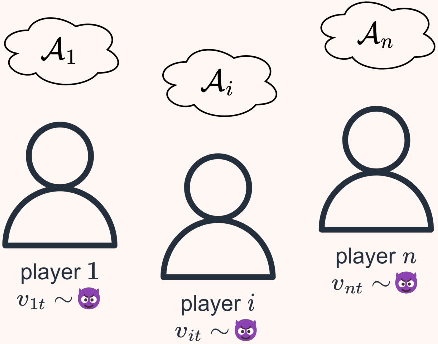
# Guarantees in Sequential First-price Auctions

**Theorem – Liquid Welfare guarantee**

If players are $\gamma$-competitive then

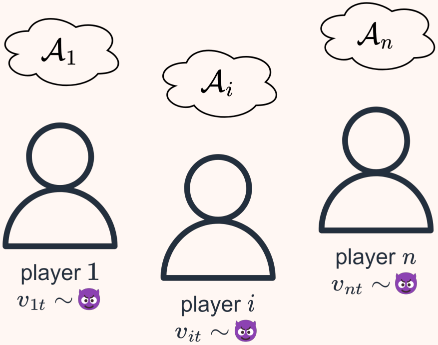
$$\mathtt{LW} \geq \frac{\mathtt{LW}^* - O(n\mathtt{Reg})}{\gamma + \frac{1}{2} + O\left(\frac{1}{\gamma}\right)}$$

Denominator is 2.41 when $\gamma = 1$




$$\forall i,\ U_i \geq \frac{\max_\lambda \hat{U}_i(\lambda) - \mathtt{Reg}}{\gamma}$$

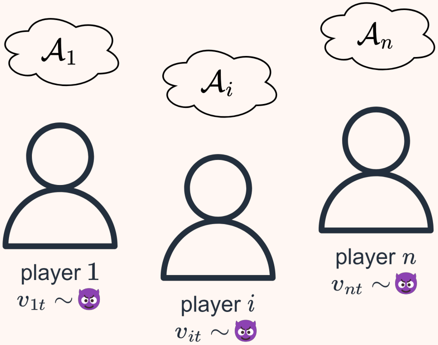
# Guarantee Intuition




$$\forall i,\ U_i \geq \frac{\max_\lambda \hat{U}_i(\lambda) - \mathrm{Reg}}{\gamma}$$

# Guarantee Intuition

1. $\hat{U}_i(\lambda)$ is “high”

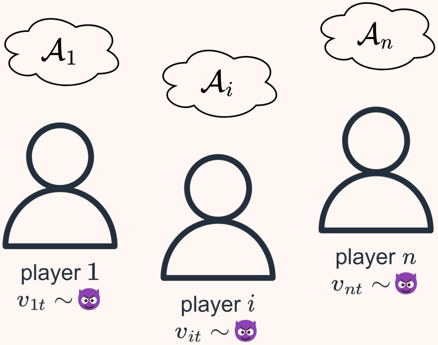



$$\forall i,\ U_i \geq \frac{\max_\lambda \hat{U}_i(\lambda) - \mathrm{Reg}}{\gamma}$$

# Guarantee Intuition

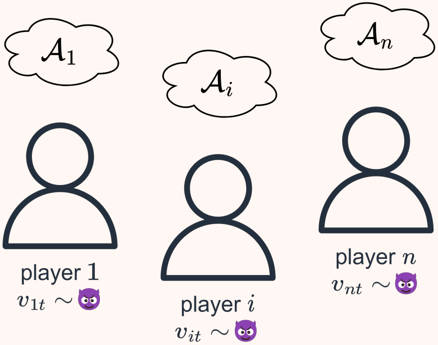
1. $\hat{U}_i(\lambda)$ is "high"
   - If $\lambda$ runs out of budget:

$$\hat{U}_i(\lambda) \geq \left(\frac{1}{\lambda} - 1\right) B_i$$



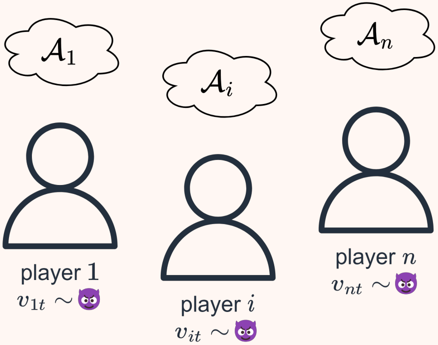

$$\forall i,\ U_i \geq \frac{\max_\lambda \hat{U}_i(\lambda) - \text{Reg}}{\gamma}$$

1. $\hat{U}_i(\lambda)$ is "high"
   - If $\lambda$ runs out of budget:
   $$\hat{U}_i(\lambda) \geq \left(\frac{1}{\lambda} - 1\right) B_i$$
   - If $\lambda$ does not run out of budget:
   $$\hat{U}_i(\lambda) \geq f(\lambda)\mathtt{LW}_i^* - g(\lambda) \sum_{t \in O_i} p_t$$




$$\forall i,\ U_i \geq \frac{\max_\lambda \hat{U}_i(\lambda) - \mathrm{Reg}}{\gamma}$$

1. $\hat{U}_i(\lambda)$ is "high"
   - If $\lambda$ runs out of budget:
     $$\hat{U}_i(\lambda) \geq \left(\frac{1}{\lambda} - 1\right) B_i$$
   - If $\lambda$ does not run out of budget:
     $$\hat{U}_i(\lambda) \geq f(\lambda)\mathtt{LW}_i^* - g(\lambda) \sum_{t \in \mathcal{O}_i} p_t$$
2. $\mathtt{LW}_i$ is "high"




$$\forall i,\ U_i \geq \frac{\max_\lambda \hat{U}_i(\lambda) - \mathrm{Reg}}{\gamma}$$

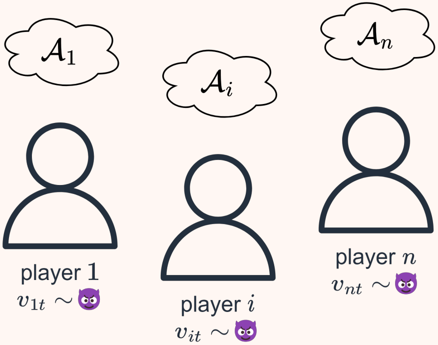
1. $\hat{U}_i(\lambda)$ is "high"
   - If $\lambda$ runs out of budget:

$$\hat{U}_i(\lambda) \geq \left(\frac{1}{\lambda} - 1\right) B_i$$

   - If $\lambda$ does not run out of budget:

$$\hat{U}_i(\lambda) \geq f(\lambda)\mathtt{LW}_i^* - g(\lambda) \sum_{t \in \mathcal{O}_i} p_t$$

2. $\mathtt{LW}_i$ is "high"
   - If $V_i \leq B_i$: utility bound




$$\forall i,\ U_i \geq \frac{\max_\lambda \hat{U}_i(\lambda) - \mathrm{Reg}}{\gamma}$$

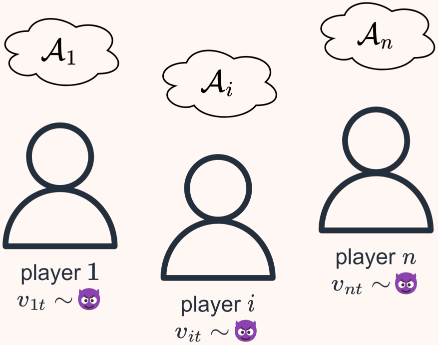
1. $\hat{U}_i(\lambda)$ is "high"
   - If $\lambda$ runs out of budget:

$$\hat{U}_i(\lambda) \geq \left(\frac{1}{\lambda} - 1\right) B_i$$

   - If $\lambda$ does not run out of budget:

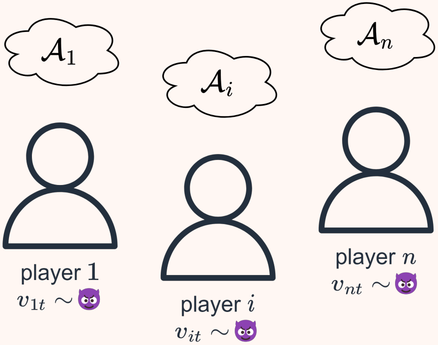
$$\hat{U}_i(\lambda) \geq f(\lambda)\mathtt{LW}_i^* - g(\lambda) \sum_{t \in \mathcal{O}_i} p_t$$

2. $\mathtt{LW}_i$ is "high"
   - If $V_i \leq B_i$: utility bound
   - If $V_i > B_i$: $\mathtt{LW}_i = B_i$




$$\forall i,\ U_i \geq \frac{\max_\lambda \hat{U}_i(\lambda) - \mathtt{Reg}}{\gamma}$$

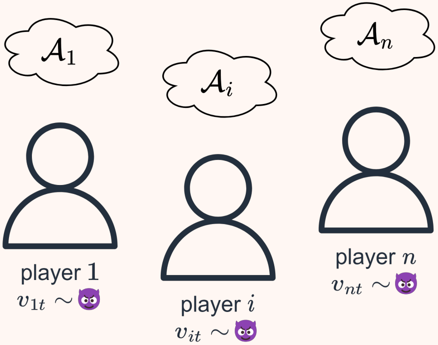
For any $\gamma \geq 1$ even if




$$\forall i,\ U_i \geq \frac{\max_\lambda \hat{U}_i(\lambda) - \mathrm{Reg}}{\gamma}$$

For any $\gamma \geq 1$ even if

- $n = 2$
- $v_{it} = v_i$
- players are $\gamma$-competitive, $\mathrm{Reg} \leq 1$



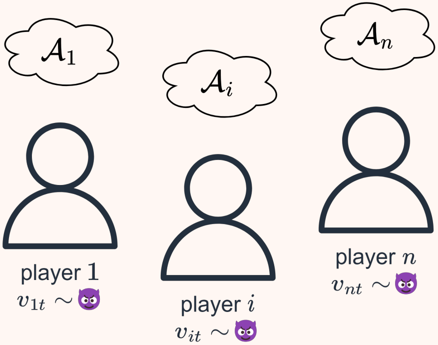

$$\forall i,\ U_i \geq \frac{\max_\lambda \hat{U}_i(\lambda) - \mathrm{Reg}}{\gamma}$$

For any $\gamma \geq 1$ even if

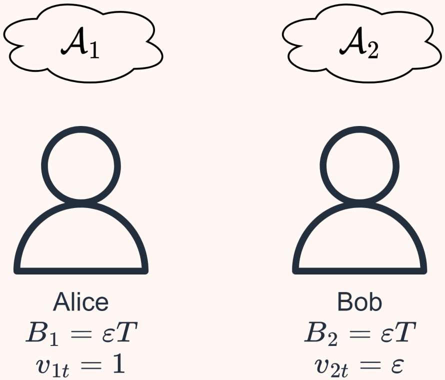
- $n = 2$
- $v_{it} = v_i$
- players are $\gamma$-competitive, $\mathtt{Reg} \leq 1$

it can hold

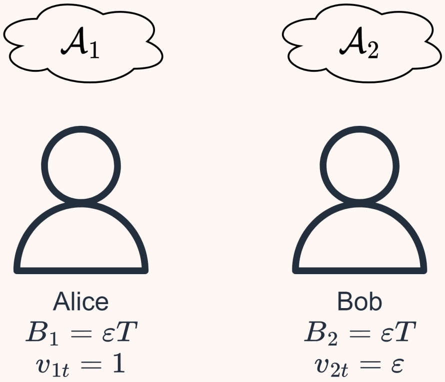
$$\mathtt{LW} \leq \frac{\mathtt{LW}^*}{\max\{\gamma, 2\}}$$




$$\forall i,\ U_i \geq \frac{\max_\lambda \hat{U}_i(\lambda) - \mathtt{Reg}}{\gamma}$$

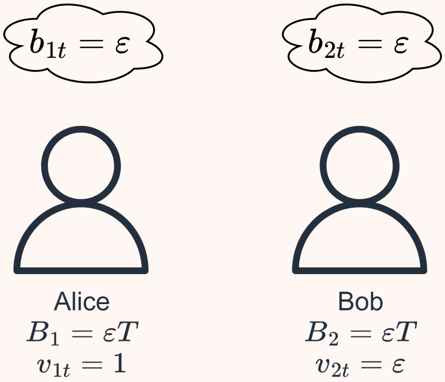

$\mathcal{A}_1$
$\mathcal{A}_2$
Alice
$B_1 = \varepsilon T$
$v_{1t} = 1$
Bob
$B_2 = \varepsilon T$
$v_{2t} = \varepsilon$

- $LW^* \approx 2\varepsilon T$

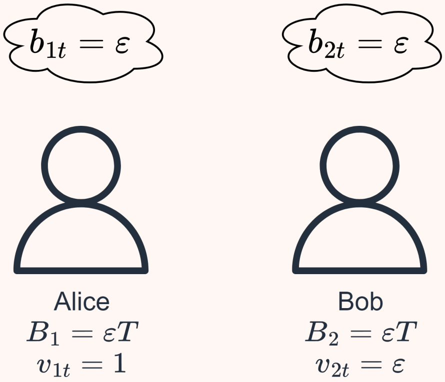


# 2 PoA Bound

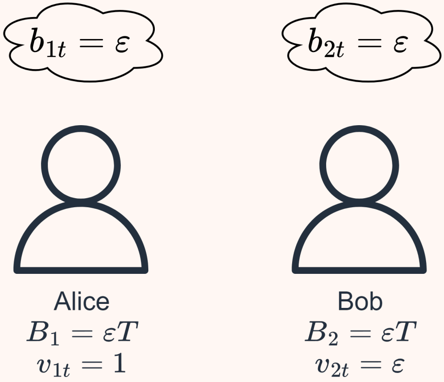
- $LW^* \approx 2\varepsilon T$



- $LW^* \approx 2\varepsilon T$
- $\forall i,\ U_i = \max_\lambda \hat{U}_i(\lambda)$



- $LW^* \approx 2\varepsilon T$
- $\forall i,\ U_i = \max_\lambda \hat{U}_i(\lambda)$
- $LW = \varepsilon T$



**Theorem**

If players are $\gamma$-competitive with submodular valuations then

$$\texttt{LW} \geq \frac{\texttt{LW}^* - O(n\mathrm{Reg})}{\gamma + 1 + O\left(\frac{1}{\gamma}\right)}$$

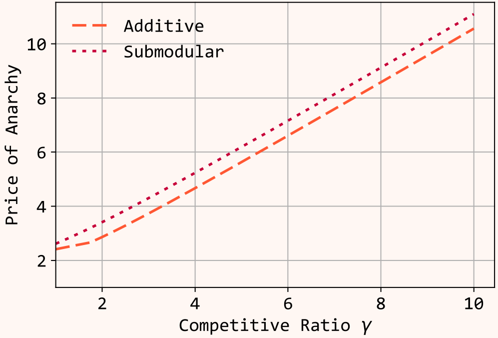
Denominator is 2.62 when $\gamma = 1$.

**Theorem**

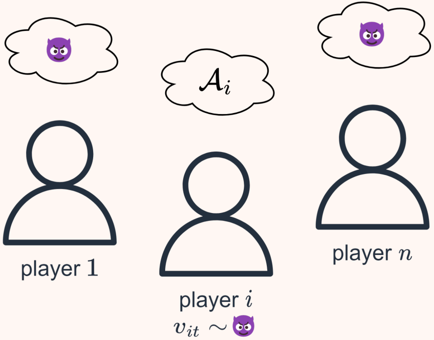
If players are $\gamma$-competitive with submodular valuations then

$$\mathtt{LW} \geq \frac{\mathtt{LW}^* - O(n\mathrm{Reg})}{\gamma + 1 + O\left(\frac{1}{\gamma}\right)}$$

Denominator is 2.62 when $\gamma = 1$.



# Individual Player Guarantees for Additive valuations



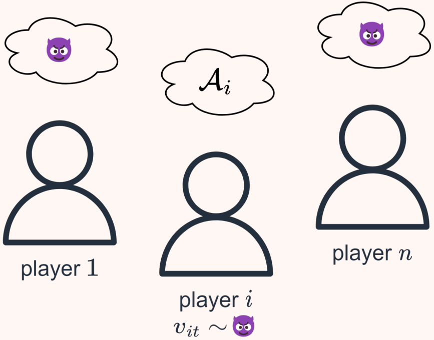
# Individual Player Guarantees for Additive valuations

**Theorem**

Player $i$ can guarantee with high probability

$$U_i \geq \frac{\max_\lambda \hat{U}_i(\lambda) - \tilde{O}(\sqrt{T})}{T/B_i}$$

if $B_i = \Omega(T)$.



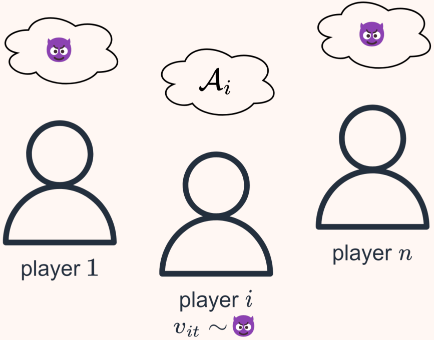
**Theorem**

Player $i$ can guarantee with high probability

$$U_i \geq \frac{\max_\lambda \hat{U}_i(\lambda) - \tilde{O}(\sqrt{T})}{T/B_i}$$

if $B_i = \Omega(T)$.

- Matches $\mathcal{A}_{AP}$
- Best possible



- Second-price:
  1-competitive $\not\Longrightarrow$ bounded PoA
- First-price:
  $\gamma$-competitive $\Longrightarrow$ PoA $\leq \gamma + \frac{1}{2} + O\left(\frac{1}{\gamma}\right)$
- PoA $\geq \max\{\gamma, 2\}$
- Submodular: PoA $\leq \gamma + 1 + O\left(\frac{1}{\gamma}\right)$
- Additive players can be $\frac{T}{B}$-competitive